Urszula Bentkowska, Paweł Drygaś,
Anna Król, Barbara Pękala, Ewa Rak (Eds.)

# International Symposium on Fuzzy Sets

## Uncertainty Modelling

## ISFS 2023

## Abstracts


University of Rzeszów
Rzeszów, Poland, May 19-21, 2023

**Urszula Bentkowska, Paweł Drygaś,**
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*Editors*

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Rzeszów 2023

ISBN 978-83-8277-077-3

All papers have been reviewed

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# Preface

**The International Symposium on Fuzzy Sets (ISFS) is organized by Polish Society for Fuzzy Sets (POLFUZZ) and University of Rzeszów in cooperation with the Faculty of Civil Engineering, Slovak University of Technology (STU) in Bratislava.**

The conference will provide an excellent international forum for sharing knowledge and results in theory, methodology, and applications of fuzzy sets and systems. It will establish a platform for discussion of critical research environments and disciplines as well as indicate the need for changes in the cooperation of the scientific community and with business partners. Research presentations are on the following topics:

- Theoretical foundations of fuzzy logic and fuzzy set theory;
- Imprecise information modeling with fuzzy, rough, and other methods;
- Federated learning;
- Image processing and computer vision;
- Information retrieval;
- Knowledge representation and engineering;
- Decision-making models;
- Expert systems;
- Intelligent data analysis and data mining;
- Approximate reasoning.
- Medical and healthcare systems;
- Business process modeling;
- Social and economic models.

The idea to organize a conference in Rzeszów aimed at integrating the local and international environment dealing with fuzzy sets and related topics was born in 2014. A group risen chaired by Professor Józef Drewniak, composed of Urszula Bentkowska, Paweł Drygaś, Anna Król, Barbara Pękala, and Ewa Rak, took the initiative to organize an international conference at the University of Rzeszów with the topic of fuzzy set theory, its extensions, and applications. Organized in 2023, ISFS is the fifth edition of this conference and the first conference organized under the auspices of POLFUZZ.

The honorary patronage of the ISFS conference is traditionally taken by the Rector of University of Rzeszów Professor Sylwester Czopek. The organizers would also like to thank for support the Vice-Rector of College of Natural Sciences UR Professor Idalia Kasprzyk.

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# Plenary speakers

## Radko Mesiar

Prof. Radko Mesiar received the Graduate degree from the Faculty of Mathematics and Physics and the Ph.D. degree with the thesis „Subadditive martingale processes“ from Comenius University, Bratislava, Slovakia, in 1974 and 1979, respectively. In 1996, he received the D.Sc. degree from the Academy of Sciences of the Czech Republic, Prague, Czech Republic.

Since 1978, he has been a Member of the Faculty of Civil Engineering, Department of Mathematics, Slovak University of Technology (STU), Bratislava, where he is currently a Full Professor. In 1983, he became an Associate Professor and, in 1998, a full Professor. Since 1994, with small interruptions, he is a head of his department. In 1995, he became a Fellow Member of the Czech Academy of Sciences, Institute of Information and Automation, Praha, Czech Republic, and, in 2006, of IRAFM, University of Ostrava, Ostrava, Czech Republic.

He is the co-author of two influencial scientific monographs: Triangular Norms (Kluwer, Dordrecht, 2000) and Aggregation Functions (Cambridge,U.K.: Cambridge Univ. Press, 2009), and co-editor of ten edited volumes. He is the (co-)author of more than 410 papers in impacted WoS journals and about 230 other papers in WoS, where his h-index is now about 50. His research interests include aggregation operators, measure and integral theory, uncertainty modelling, fuzzy sets and fuzzy logic, multicriteria decision support (including scintometric applications), copulas, triangular norms, and intelligent computing.

Prof. Mesiar has been elected as a Fellow of the International Fuzzy Systems Association since 2011 and awarded as a brilliant scientist by EUSFLAT association in 2017. Since 2020 he is a honorary member of EUSFLAT. He is a Founder and Organizer of the FSTA, ISCAMI, ISAS, and AGOP international conferences.

He is an advisory editor of distinguished journals Fuzzy Sets and Systems and Information Sciences, and member of editorial board of International Journal of Approximate Reasoning, International Journal of General Systems, Kybernetika, Iranian Journal of Fuzzy Systems, Axioms, and Granular Computing, among others.

## Susana Montes

Susana Montes received the M.Sc degree in mathematics, option statistics and operational research, from the University of Valladolid, Valladolid, Spain, in 1993, and the Ph.D. (cum laude) degree from the University of Oviedo, Gijón, Spain, in 1998. Dr. Montes received the Best Mathematics Ph.D. Thesis Award from the University of Oviedo.

She is currently a Full Professor with the area of Statistics and Operational Research, University of Oviedo, where she is the Leader of the official research group UNIMODE and she is also the head of the Department of Statistics and Operational Research and Didactic of Mathematics. She has 86 publications in JCR journals, 36 book chapters and 148 communications, 96 of them in international conferences. She has participated in national (15) and international (3) projects at the moment, some of them led by her (6 national and 1 international). Moreover, she was the leader of 5 contracts with companies and she has been a member of the work team for other 7 contracts. She was the Treasurer of EUSFLAT from 2015 to 2019, she was the secretary of this society from 2019 to 2021 and from 2021 she is the president. Apart from that, she is also Vice-president of the international society IFSA. She has been an Associate Editor of the journals: IEEE Transactions on Fuzzy Systems, Journal of Intelligent and Fuzzy Systems, Mathematics, Computational and Applied Mathematics and the Electronic Journal of Applied Statistical Analysis, as well as, she has been invited editor twice in Fuzzy Sets and Systems, once in International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems and once in Information Sciences. She has supervised 5 thesis, the last one in 2019. As a consequence of the previous merits, she has been granted with four research periods and one transfer period from the National Commission for Assessment of Research Activity.

## Janusz Kacprzyk

Janusz Kacprzyk is Professor of Computer Science at the Systems Research Institute, Polish Academy of Sciences, WIT – Warsaw School of Information Technology, and Chongqing Three Gorges University, Wanzhou, Chongqing, China, and Professor of Automatic Control at PIAP – Industrial Institute of Automation and Measurements in Warsaw, Poland. He is Honorary Foreign Professor at the Department of Mathematics, Yli Normal University, Xinjiang, China. He is Full Member of the Polish Academy of Sciences, Member of Academia Europaea, European Academy of Sciences and Arts, European Academy of Sciences, Foreign Member of the: Bulgarian Academy of Sciences, Spanish Royal Academy of Economic and Financial Sciences (RACEF), Finnish Society of Sciences and Letters, Flemish Royal Academy of Belgium of Sciences and the Arts (KVAB), National Academy of Sciences of Ukraine and Lithuanian Academy of Sciences. He was awarded with 6 honorary doctorates. He is Fellow of IEEE, IET, IFSA, EurAI, IFIP, AAIA, I2CICC, and SMIA. His main research interests include the use of modern computation computational and artificial intelligence tools, notably fuzzy logic, in systems science, decision making, optimization, control, data analysis and data mining, with applications in mobile robotics, systems modeling, ICT etc. He authored 7 books, (co)edited more than 150 volumes, (co)authored more than 650 papers, including ca. 150 in journals indexed by the WoS. He is listed in 2020 and 2021 "World's 2% Top Scientists" by Stanford University, Elsevier (Scopus) and ScieTech Strategies and published in PLOS Biology Journal. He is the editor in chief of 8 book series at Springer, and of 2 journals, and is on the editorial boards of ca. 40 journals. He is President of the Polish Operational and Systems Research Society and Past President of International Fuzzy Systems Association.

## Józef Drewniak

Józef Drewniak was born in 1945. He received the M.Sc. and Ph.D. degrees in mathematics from the University of Silesia, Katowice, Poland, in 1969 and 1975, respectively. He received the D.Sc. degree in mathematics from the A. Mickiewicz University, Poznań, Poland, in 1990. From 1969 to 1976, he worked in the Institute of Theoretical and Applied Infomatics, Gliwice. Next, from 1976 to 2001, he worked at the Silesian University, Katowice, and from 2001 to 2016, at the University of Rzeszów as an academic teacher. Now he is retired. His research interests include convergence of iterative sequences, lattices and ordered semigroups, fuzzy sets, fuzzy numbers, fuzzy relations and fuzzy relation equations. He is the author or coauthor of more than 70 journal papers and over 20 conference presentations. He is a regular reviewer for many international journals and conferences. Moreover, he is an author over 300 reviews published in Zentrallblatt MATH.

# Table of Contents

# Aggregation functions

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Aggregation functions can be dated back almost 4000 years — recall, for example, the $14^{\text{th}}$ problem of the Moscow papyrus from about 1850 BC calculating the volume of a frustum (a truncated square pyramid). Here, we sketch the history of the roots of aggregation theory, which can be considered as a genuine research field the last 40 years. To stay on a general level, recall that, for a bounded poset $(P, \leq, 0, 1)$, and $n \in \mathbb{N}$, a mapping $A\colon P^n \to P$ is called an ($n$-ary) aggregation function (on $P$) if it is increasing in each variable and preserves the bounds, i.e., $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$, i.e., $A$ is an order homomorphism. The most common scales $P$ are the real unit interval $[0, 1]$ and finite chains $C_k = \{1, \ldots, k\}$, $k \in \mathbb{N}$, equipped with the classical ordering of reals. If $P$ is not bounded, e.g., if $P = [0, \infty[$, some modifications of the boundary conditions are considered, see, e.g., [7].

In this contribution, we first introduce the classification of aggregation functions (conjunctive, disjunctive, averaging and mixed). For example, $A\colon P^n \to P$ is a conjunctive aggregation function whenever, for all $\mathbf{x} \in P^n$, $A(\mathbf{x}) \leq x_i$ for all $i \in \{1, \ldots, n\}$. Then, some important classes of aggregation functions, including their history and some basic results, will be discussed. For example, triangular norms (t-norms for short) are aggregation functions $T\colon P^2 \to P$ which are commutative, associative and $e = 1$ is their neutral element. They are dated back to Schweizer and Sklar [13] in the case $P = [0, 1]$, though, for several results concerning t-norms, the works from semigroups theory by Clifford [3], Mostert and Shields [11], or Birkhoff [1] could be considered.

Considering the $n$-ary extension $T^{(n)}$ of a t-norm $T$, $T^{(n)}\colon P^n \to P$, $n > 2$, we can characterize $n$-ary t-norms by the Post associativity [12]. In the case of $n$-ary t-norms , the neutral element $e = 1$ means that $T^{(n)}(x_1, \ldots, x_n) = x_i$ whenever $x_j = 1$ for each $j \neq i$. Among the related aggregation functions, we will recall triangular conorms, uninorms and nullnorms, including some interesting examples.

These types of aggregation functions are strongly related to many-valued logics. For example, considering a 3-valued logic, i.e., for $P = \{0, 1/2, 1\}$, see Łukasiewicz [9], there are only 2 t-norms, $T = T_{\mathrm{M}} = \min$, and $T = T_{\mathrm{L}}$ characterized by $T_L(1/2, 1/2) = 0$, thus leading to 2 types of 3-valued logics.

Aggregation functions closely related to t-norms are also copulas, quasi-copulas and semicopulas [6]. Recall that a copula $C\colon [0, 1]^n \to [0, 1]$ expresses the stochastic dependence structure of an $n$-ary random vector $X = (X_1, \ldots, X_n)$,

$$F_X(x_1, \ldots, x_n) = C\Big(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\Big),$$

where $F_X$ is the joint distribution function of the random vector $X$, and $F_{X_i}$ are the corresponding marginal distribution functions.

Important aggregation functions related to decision problems, primarily to multi-criteria decision support, are different kinds of means (averaging aggregation functions). Considering $P = [0, 1]$, we recall aggregation functions related to the arithmetic mean, e.g., (weighted) quasi-arithmetic means, OWA operators, the Choquet integrals, etc. For more details see [7]. As an interesting example we recall the mean introduced by Hero of Alexandria (also known as Heron of Alexandria) around 60 AD. His mean is obtained as a composite of 3 means, namely, the weighted arithmetic mean with weights $2/3$ and $1/3$, the arithmetic mean and the geometric mean, i.e., $H = W_{2/3,1/3}(AM, AG)$, and it solves a generalized form of the mentioned problem from the Moscow papyrus.

Further, some other types of integrals will be presented, including copula-based integrals and semi-copula based integrals. For example, the known Choquet integral [2] corresponds to the product copula, i.e., the copula of independence, and the copula of total positive dependence $C = \min$ generates the Sugeno integral [15]. As a semi-copula based integral we recall, e.g., the Shilkret integral [14]. We also recall some algebraic approaches leading to new constructions of aggregation functions such as the Galois connection [5] or the Möbius transform [10].

Our next aim is to discuss aggregation functions in the framework of sufficiently specific but still rather general lattices or posets, see, e.g., [4]. If finite chains are considered, only some aggregation functions acting on $[0, 1]$ can be extended to these new scales. We recall some of them. We will also consider aggregation functions on product lattices and on distributive bounded lattices. As a surprising result obtained in this more general framework we recall the characterization of the Sugeno integrals on distributive lattices [8] as aggregation functions which preserve congruences. Note that the corresponding result for the Sugeno integral in the original case, i.e., for $P = [0, 1]$, was not known before.

**Acknowledgements**
The author was supported by the Slovak Research and Development Agency under the contract no. APVV-18-0052.

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# Subsethood and embedding measures for interval-valued fuzzy sets: Similarities and differences

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One of the main extensions of the fuzzy sets are the interval-valued fuzzy sets [6]. They were introduced independently by Zadeh [12], Grattan-Guiness [8], Jahn [9] and Sambuc [11] in the seventies, in order to treat jointly with vagueness and uncertainty. Nowadays a lot a studies are devoted to this concept. Since the membership value in this case is an interval, comprehending and interpreting this ambiguity is crucial when working with this structure. In our particular context, we examine interval-valued fuzzy sets from an epistemic perspective [7], where the actual membership value is a specific number that is unknown to us, but we possess knowledge of a range of possible values. Thus, the study of the set of closed subintervals of the interval $[0, 1]$ is a fundamental step in this field.

Based on this idea, we will start by considering a subsethood measure for intervals. From an axiomatic definition, we will be able to construct these measures based on the width of the interval and also based on implication functions [1]. The aggregation of these values allow us to obtain a new measure, called IV-embedding, which is a novel metric for comparing two interval-valued fuzzy sets by means of their precision [3]. Due to its importance, this concept is subject to an in-depth investigation aimed at understanding its principal properties.

It is worth noting that although a subsethood measure for intervals provides a measure of the embedding degree of one interval-valued fuzzy set within another, it does not yield a measure of the degree of inclusion [4, 10]. While this is reasonable given the concepts of embedding and inclusion, it may not be entirely intuitive. Thus, it is necessary to change the starting point, that is, the subsethood measure to a more general concept in order to be able to generate subsethood measure for interval-valued fuzzy sets. This is done by considering the typical orders for intervals generalizing the usual order in the real line instead of the inclusion. Just changing the considered order, we can generate an inclusion measure on the set of interval-valued of fuzzy sets by considering, for instance, admissible orders [5]. This is not a simple change, since it is clear that it changes totally the information we are analysing.

Additionally, these measures between intervals based on general orders can also be used to determine the degree of similarity between two intervals by combining the degree to which each interval is smaller than the other through an increasing and one-strict aggregation function [2]. Finally, a decomposable similarity measure for interval-valued fuzzy sets [13] can be obtained by combining the similarity between the intervals that define the degree of membership at each point for the elements of a finite referential. We can also obtain a similarity measure for interval-valued fuzzy sets from a inclusion measure. Both approaches for obtaining similarity measures are compared.

In summary, this talk will take us on a tour through various structures related to interval-valued fuzzy sets. Specifically, we will examine the similarities, differences and connections between embedding measures, subsethood measures and similarity measures.

**Acknowledgements**
The work was supported from the Spanish Ministry of Science and Technology projects PGC2018-098623-B-I00.

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# De novo type approaches in fuzzy decision making, optimization and control: Towards optimal system design

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We are concerned with the so-called fuzzy optimization meant as optimization, static and dynamic, under imprecise (fuzzy) information as to the goals (performance or objective functions) and constraints. The essence of (fuzzy) optimization is usually meant as to find optimal values of some variables that maximize or minimize some value (maybe utility) function exemplified by a financial result of some activities subject to some constraints on, for instance, available resources. This traditional formulation presupposes that some problem specifications as, e.g., right hand sides in constraints on resources devoted to particular types of activities are given in advance. This general setting has been very successful for solving a rich variety of problems for decades.

However, one can see at the first glance that the above setting of optimization problems boils down to an optimal allocation of fixed or limited resources that are given in advance. This may be in contrast to many, if not most, real world practical problems that are rather meant to (optimally) design a particular system. For instance, to improve or futher develop transportation system we can be interested in its proactive extension by, e.g.,assigning some resources for increasing total capacities of passengers or freight at sources and destination by, for instance, building some additional facilities like stations, under a total financial limitation. That is, the right hand sides, roughly speaking, can be not prespecified in advance, and are subject to optimization.

The above idea of an optimal system design, the so-called de novo programming, was introduced by Zeleny in the very early 1980s, first for single objective problems, then extended for multiobjective problems, and then for fuzzy dynamic programming, and had been applied to solve many practical problems. The approach had become popular in fuzzy optimization too.

This de novo programming approach will be presented for both the static and dynamic problem formulation, and illustrated on examples.

# Algebra of fuzzy relations

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Mathematical theory of binary relations was developped by De Morgan [4], Schröder [17], Russell [16], Tarski [19] and Schwarz [18]. Independently, an algebra of Boolean matrices was dicussed e.g. by Luce [13], Blyth [3], Rudeanu [15] and Kim [11].

After introduction of fuzzy relations by Zadeh [21] the theory of fuzzy relations was developed. It is provided in papers and books by e.g. Goguen [8], Zadeh [22], Kaufmann [10], Drewniak [5], Fodor, Roubens [7], Bandemer, Gottwald [1], Bělohlávek [2], Peeva, Kyosev [14] and many other.

Usually theoretical considerations of fuzzy relations are followed by examples with matrices of fuzzy relations on a finite set. However many papers concerns directly matrices of fuzzy relations called then by fuzzy matrices cf. e.g. Thomason [20], Hashimoto [9], Li [12] or Fan [6].

We have observed here some similarities and differences in comparison with the theory of binary relations and with the algebra of Boolean matrices.

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# On the characterizations of the Reichenbach implication by functional equations

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The Reichenbach implication $I_{RC}(x, y) = 1 - x + xy$ is an important implicational/conditional function in many-valued logic and probability theory. Although it is well-known that the Reichenbach implication can be generated in different ways, including the construction ways of $(S, N)$-implication, $(T, N)$-implication, $f$-generated implication and quantum logic implication (see [1]), its analytic characterization had remained open since its birth in 1935 until Massanet and Torrens gave one characterization in [2] in terms of the law of importation and the standard negation following the generating way of $f$-generated implications. In this talk, we will provide several new characterizations for the Reichenbach implication using the migrativity equation, two distributivity equations and some other algebraic properties, respectively, of which some are independent of any requirements on its natural negation. Moreover, we will show a new representation of $f$-generated implications by transforming the Yager implication, which contains, as a particular case, a new representation of the Reichenbach implication. All these new results are presented with the proofs in [3].

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# The generalized Choquet integral computation in discrete space

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The theory of nonadditive measures and integrals was significantly developed in the last decades starting with the publication of Vitali [5] in 1925 and following the work of Gustave Choquet [2]. During that epoch, several nonadditive integrals, that are well-known today, were defined. Among them is the Choquet integral that is the improper Riemann integral of the survival function of a vector $\mathbf{x} = (x_1, \dots, x_n) \in [0,\infty)^{[n]}$ with respect to a monotone measure $\mu\colon 2^{[n]} \to [0,\infty)$, i.e.

$$\mathrm{Ch}(\mathbf{x},\mu) := \int_0^\infty \mu(\{\mathbf{x} > \alpha\})\,\mathrm{d}\alpha,$$

where $\mu(\{\mathbf{x} > \alpha\}) := \mu\left(\{i \in [n] : x_i > \alpha\}\right)$, $\alpha \in [0,\infty)$ and $[n] := \{1, 2, \dots, n\}$, $n \geq 1$, $n \in \mathbb{N}$. This concept is applicable in many (real) areas of everyday life, see [3, 4]. It is mainly thanks to the monotone measure $\mu$, which takes into account the interaction between the input data. It is known that the Choquet integral can be rewritten in a discrete space as follows

$$\mathrm{Ch}(\mathbf{x},\mu) := \sum_{i=0}^{n-1} (x_{(i+1)} - x_{(i)}) \cdot \mu\left(E_{(i+1)}\right), \tag{1}$$

with the permutation $(\cdot)$ such that $0 = x_{(0)} \leq x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and $E_{(i)} = \{(i), \dots, (n)\}$ for $i \in [n]$.

There exist a lot of generalizations of the Choquet integral. One of them was introduced in [1]. The generalization is based on the rewritting the survival function as follows

$$\mu\left(\{i \in [n] : x_i > \alpha\}\right) = \min\left\{\mu(E^c) : \max_{i \in E} x_i \leq \alpha,\ E \in 2^{[n]}\right\}$$

for any $\alpha \in [0,\infty)$. Replacing the maximum by the conditional aggregation operator and the power set by its subset we get the generalized survival function:

$$\mu_{\mathscr{A}}(\mathbf{x},\alpha) := \min\left\{\mu(E^c) : \mathsf{A}(\mathbf{x}|E) \leq \alpha,\ E \in \mathscr{E}\right\}, \quad \alpha \in [0,\infty),$$

with $\mathscr{A} = \{\mathsf{A}(\cdot|E) : E \in \mathscr{E} \subseteq 2^{[n]}\}$ being a family of conditional aggregation operators. Motivated by the standard Choquet integral, one can define the so-called generalized Choquet integral as a improper Riemann integral of the generalized survival function, i.e.

$$\mathrm{C}_{\mathscr{A}}(\mathbf{x},\mu) := \int_0^{\infty} \mu_{\mathscr{A}}(\mathbf{x},\alpha)\,\mathrm{d}\alpha,$$

see [1]. The generalized Choquet integral is a new concept. It is not known any formula for its calculation analogous to formula (1). In this contribution, we derive many computational formulas for the generalized Choquet integral and we point out different approaches to these constructions. We supplement the above-mentioned results with graphic visualizations and pseudo-algorithms. Using a practical example, we point out the need of studying concepts of conditional aggregation operators and integration with respect to them.

**Acknowledgements**
The work was supported by the grant scheme VVGS-PF-2022-2143, and the grants APVV-21-0468, VEGA 1/0657/22.

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# Ensemble regression model based on aggregation functions

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According to the literature most of works on ensemble learning focus on classification problems. However, approaches that are successful for classification are often not directly applicable for regression [5]. In our ensemble regression model we apply aggregation functions (cf. [2]) since they proved to be a useful tool in machine learning models and ensemble learning methods [3]. Ensemble learning is applied to provide better predictive performance by combining the predictions from a collection of input base models (cf. [4, 6]). We consider microarray datasets which consist of large number of features and small number of instances.The presented ensemble model is based on the method of modification of the input features set. The predictions delivered by the group of ensemble regression models are then combined using aggregation functions. In such model we apply aggregation functions defined on arbitrary interval $[a, b]$ (cf. [1]) such as quasi-arithmetic means, ordered weighted averages, some well-known convex combinations of means. We provide a discussion of aggregation functions impact on regression quality.

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# The Choquet integral in the language of bases and transforms

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In the literature, see [3] or [4], one can find several representations of well-known Choquet integral via various bases of set functions and their corresponding transforms (e.g. Möbius, co-Möbius, Fourier transform). Recall that if $\Psi\colon \mathbb{R}^{2^X} \to \mathbb{R}^{2^X}$ is a linear invertible transform with $\mu \mapsto \Psi_\mu$, then by [2, Lemma 2.91] there exists a unique basis $\mathrm{B} = (b_A^\Psi)_{A\subseteq X}$ of set functions such that for any set function $\mu\colon 2^X \to \mathbb{R}$ we have

$$\mu = \sum_{A\subseteq X} \Psi_\mu(A) \cdot b_A^\Psi. \tag{1}$$

The statement holds also conversely, i.e., for every basis B, there is a unique transform $\Psi$ such that (1) is true.

From this fact and the linearity of the Choquet integral w.r.t. a game[3] we get the formula

$$\int f \mathrm{d}\mu = \sum_{A\subseteq X} \Psi_\mu(A) \cdot \int f \mathrm{d}b_A'^\Psi, \tag{2}$$

where $(b_A'^\Psi)_{A\subseteq X}$ is the set of games and differs from $(b_A^\Psi)_{A\subseteq X}$ only on empty set. More precisely, since $b_A^\Psi(\emptyset) = 0$ does not hold for any $A \subseteq X$, in general[4], we put $b_A'^\Psi(\emptyset) = 0$ for any $A \subseteq X$.

In our contribution, we discuss formula (2) in comparison with equation (4.54) in Grabisch book from 2016. Indeed, formula (4.54) erroneously excludes the empty set, see [1], which has important consequences for deriving Choquet integral representations in the literature using this approach. On the one hand, several results obtained from (4.54) are not affected for some transforms, e.g. the co-Möbius transform. On the other hand, several formulas should be corrected, e.g. for the Fourier transform. Lastly, a new formula for the Choquet integral representation via the Shapley interaction transform will be provided.

[3] A set function $\mu\colon 2^X \to \mathbb{R}$ is a *game* if $\mu(\emptyset) = 0$.
[4] An example of basis that is nonzero in empty set is $b_A^\Psi(C) = (-1)^{|A\cap C|}$, which correspond to the Fourier transform.

**Acknowledgements**
The work was supported by the Slovak Research and Development Agency under the contract No. APVV-21-0468, the grant VEGA 1/0657/22, and the internal scientific grant vvgs-pf-2022-2143.

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# Postmodifiers preserving monotony

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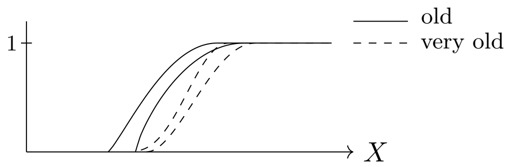
Any fuzzy set could represent a linguistic variable, for example "old". Another value "very old" is a kind of a modified variable. There are also other applications of fuzzy sets, for example in decision theory, where in a model of expert rating modifications may play some role.




In our case a modifier $m$ is a transformation of a set to another set. The research is focused on classical fuzzy sets ($A : X \to [0,1]$) and lattice-valued fuzzy sets ($A : X \to L$), specifically on interval-valued fuzzy sets ($A : X \to L([0,1])$), shortly IVFSs.

If it is possible to express $m$ as $m = r \circ A$ we speak about a modifier with pure postmodification and $r$ is called a postmodifier. More about its properties can be found in [1].

The research question is when the postmodifier $r$ preserves mono-tony and convexity. We showed that increasing $r$ preserves convexity for classical fuzzy sets and meet homomorphism $r$ preserves convexity for lattice-valued fuzzy sets.

If we deal with IVFSs we are able to characterize $r$ for different orders (lattice order, lexicographical order of type 1, Xu and Yager order, interval dominance or others). More about given orders can be found for example in [2]. Every order $\preceq$ induces the corresponding $\preceq$-convexity. More about problematic definition of convexity for IVFSs can be found in [3].

**Acknowledgements**
The work was supported from the Grant No. 1/0150/21 provided by Slovak grant agency VEGA.

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# The importance of keywords in bibliometrics

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The main focus of bibliometrics is to evaluate scientific outputs in a quantitative manner, e.g. by investigating papers' citations, introducing various models capable to capture the essence of citation process, or studying the structure of the whole citation network (see e.g. [1, 2]). Another interesting and popular research theme in bibliometrics is keyword analysis. Each publication is usually equipped with several words or phrases - keywords - that represent the document topic. Keywords are often considered to be an important and condensed contents of academic publications for any discipline as well as carriers of scientific concepts, ideas, and knowledge. Publication keywords have been widely used e.g. to study the growth of research domains as well as to predict the knowledge evolution (see e.g. [3, 4]).

Of course, any bibliometric research should be based or verified on real-life bibliometric data. Nowadays, there are few open-source scientific databases available to researchers. For example, the ArXiv preprints' database (see [5]) - an open-access archive of more the 2 million scholarly articles in the fields of physics, mathematics, computer science, etc. Some of the main advantages of ArXiv are the fast publishing process that allows to increase the speed of research exchange and easy access to papers data. One has to keep in mind, however, that articles in Arxiv are not peer-reviewed. Another free data source can be found in the DBLP bibliography database [6] that consists of more than 4 million records of computer science authors and their papers as well as additional information - e.g. keywords.

The investigation carried out in this paper includes a comparison of selected open-source bibliometric databases in terms of keyword behavior. Various modeling techniques are considered: from quantitative analysis to keyword co-occurrence network modeling. In addition to the empirical analysis conducted, a review of the most interesting research trends in the area of keyword analysis was conducted.

**Acknowledgements**
The authors would like to thank dr Barbara Żogała-Siudem, dr Grzegorz Siudem and dr Marek Gagolewski for valuable feedback and remarks.

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# Solution methods for some constrained OWA aggregation problems

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In 1988 Yager introduced a new aggregation technique based on the ordered weighted averaging (OWA) operators [17], which often proved to be really useful in multi-criteria decision problems. An interesting particular problem is when we search for the optimal value of the OWA function with respect to linear constraints. This problem was again formulated by Yager in [15]. An interesting open question is to find an analytical solution to this nonlinear programming problem as a function depending on the OWA weights and the coefficients in the linear constraints. In the case of a single constraint the resulting problem can be formulated (for the maximum problem) as

$$\begin{cases} \max F(x_1,\ldots,x_n) = w_1y_1 + \cdots + w_ny_n \ , \\ \alpha_1x_1 + \cdots + \alpha_nx_n \leq 1 \\ x_i \geq 0, i = \overline{1,n}, \end{cases} \tag{1}$$

where $F(x_1,\ldots,x_n) = w_1y_1 + \cdots + w_ny_n$ is the OWA function for the given nonnegative weights $w_1...,w_n$, with their sum equal to 1. It means that the vector $(y_1,...,y_n)$ is permuted from $(x_1,...,x_n)$, by rearranging the components in nonincreasing order. Problem (1) was completely solved in [2] for the special case when $\alpha_1 = \cdots = \alpha_n = 1$. Another interesting approach for this problem can be found in [1]. The idea in [2] was to obtain the solution from the dual of a linear program. This method actually works when the coefficients are arbitrary, therefore, problem (1) was completely solved in [5]. Additionally, the case when we search for the minimum instead of maximum was also solved in [4]. The use of linear programs was successfully used in other related problems as well (see [12], [11]).

Now, considering the case of two linear constraints, the finding of the analytical solution remains an open question. In the special case of two comonotone constraints (that is, the coefficients can be rearranged in increasing order by the same permutation) and when the coefficients are positive, the problem was solved in [7].

Another interesting problem is when one tries to optimize the OWA weights $w_1, w_2, ..., w_n$ with respect to some given constraint (see [9], [3], [8], [10], [13], [14]). In [6] we found the nearest weights with respect to the Euclidean norm to the given Olympic OWA weights $\left(0, \frac{1}{n-2}, ..., \frac{1}{n-2}, 0\right)$ for a given level of orness. Note that the Olympic weights as well as the orness associated to some given weights were introduced by Yager (see [17] and [16]). The solution function is piecewise linear in the variable $\alpha$ which denotes the orness level.

A promising further research is the one in which the Olympic OWA weights are replaced with other OWA-type weights satisfying a certain symmetry property, such as for example the median OWA weights. We will present a part of the results obtained in this new direction.

**Acknowledgement**

The participation of Lucian Coroianu in this Symposium was possible with the financial support given by the University of Oradea, Romania.

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# A note about almost uniformly convergence and a variation on the Egorov's theorem

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The aim of this contribution is to define an almost uniformly convergence on MV-algebra $(\mathscr{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$ given by

$$\begin{aligned} \mathscr{M} &= \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to [0,1] \\ &\quad \text{are } \mathscr{S} - \text{measurable functions}\}, \\ \mathbf{A} \oplus \mathbf{B} &= \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \\ \neg\mathbf{A} &= (1_\Omega - \mu_A, 1_\Omega - \nu_A), \end{aligned}$$

where $(\Omega, \mathscr{S})$ be a measurable space, $\mathscr{S}$ be a $\sigma$-algebra. Here the corresponding lattice group is $(\mathscr{G}, +, \leq)$ given by

$$\begin{aligned} \mathscr{G} &= \{\mathbf{A} = (\mu_A, \nu_A); \mu_A, \nu_A : \Omega \longrightarrow R \\ &\quad \text{are } \mathscr{S} - \text{measurable functions}\}, \\ \mathbf{A} + \mathbf{B} &= (\mu_A + \mu_B, \nu_A + \nu_B - 1_\Omega), \\ \mathbf{A} \leq \mathbf{B} &\Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B. \end{aligned}$$

Recall that $(\mathscr{G}, +, \leq)$ has the neutral element $\mathbf{0} = (0_\Omega, 1_\Omega)$,

$$\mathbf{A} - \mathbf{B} = (\mu_A - \mu_B, \nu_A - \nu_B + 1_\Omega)$$

and the lattice operations

$$\begin{aligned} \mathbf{A} \vee \mathbf{B} &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \\ \mathbf{A} \wedge \mathbf{B} &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) \end{aligned}$$

(see [7, 9]).

We study a connection between almost uniformly convergence of observables of MV-algebra $\mathscr{M}$ and almost uniformly convergence of random variables in classical probability space. We prove a version of Egorov's theorem for MV-algebra $(\mathscr{M}, \oplus, \odot, \neg, (0_\Omega, 1_\Omega), (1_\Omega, 0_\Omega))$, too. We inspired by definition of almost uniformly convergence introduced by B. Riečan, M. Jurečková, A. Tirpáková and R. Bartková in papers [4–6, 8] for many valued structures. Note that a family $\mathscr{F}$ of intuitionistic fuzzy events can be embedded to a suitable MV-algebra $\mathscr{M}$. The intuitionistic fuzzy sets was introduced by K. T. Atanassov in 1983 as a generalization of Zadeh's fuzzy sets (see [1–3, 10]).

**Acknowledgements**

The work was supported from the Grant VEGA 2/0122/23 and from the Operational Programme Integrated Infrastructure (OPII) for the project 313011BWH2: InoCHF - Research and development in the field of innovative technologies in the management of patients with CHF, co-financed by the European Regional Development Fund.

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# On threshold generated fuzzy implications

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Fuzzy implications are one of the main operations in fuzzy logic. For this reason many families of these connectives and their properties are examined (e.g. [1]).

In this contribution, some methods of generating fuzzy implications from given ones are presented. Examples of such constructed families are e-threshold generated implications [2] and vertical e-threshold generated implications [3] proposed by S. Massanet and J. Torrens. The first of these approach was modified later by Z.-H. Yi and F. Qin [4] to more components and was called the extended threshold generation method. In this contribution, we propose generalization of the methods that allow us to adapt better the values of fuzzy implications for specific use.

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# AI and Sport Data Analysis in Football

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AI and Sport Data Analysis in Football (Soccer) Artificial Intelligence (AI) methods in sports analysis have been widely adopted in soccer to enhance performance analysis and provide insights into various aspects of the game (cf. [1, 2]. AI techniques such as machine learning, computer vision, and natural language processing are used to analyze player and team performance, predict game outcomes, and improve tactical decision making. AI can also be used to analyze player movements, physical attributes, and tactical patterns to provide personalized training programs and injury prevention strategies (see [3, 4]). Injury prediction is another area where AI is making a big impact in soccer (see [5, 6]). Machine learning algorithms can be trained on large amounts of player data to identify patterns and risk factors for injury. This allows teams to proactively address potential injury issues and implement preventative measures. AI can also be used to monitor player workload and fatigue levels, helping to minimize the risk of injury. In this way, AI is playing a crucial role in ensuring the health and well-being of players, while also improving their performance on the field. Overall, AI is transforming the way soccer is analyzed and helping teams and players to reach new levels of performance.

**Acknowledgements**
The research is conducted in cooperation with KKS Lech Poznań football club.

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# Scoring functions on ordered qualitative scales

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In this contribution we present some scoring functions that assign numerical scores to the linguistic terms of ordered qualitative scales. They are based on the concept of ordinal proximity measure.

Let $\mathscr{L} = \{l_1, \dots, l_g\}$ be an ordered qualitative scale (OQS) , with $g \geq 2$ and $l_1 \prec \cdots \prec l_g$. An ordinal proximity measure (OPM) on $\mathscr{L}$ is a mapping that assigns an ordinal degree of proximity to each pair of linguistic terms of an OQS $\mathscr{L}$. The ordinal degrees of proximity belong to a linear order $\Delta = \{\delta_1, \dots, \delta_h\}$, with $\delta_1 \succ \cdots \succ \delta_h$. Note that the members of $\Delta$ are not numbers, but ordinal degrees.

**Definition 1 ([1]).** *An* ordinal proximity measure on $\mathscr{L}$ with values in $\Delta$ *is a mapping* $\pi : \mathscr{L} \times \mathscr{L} \longrightarrow \Delta$, *where* $\pi(l_r, l_s) = \pi_{rs}$ *represents the degree of proximity between* $l_r$ *and* $l_s$, *satisfying:*

1. Exhaustiveness*: For every* $\delta \in \Delta$, *there exist* $l_r, l_s \in \mathscr{L}$ *such that* $\delta = \pi_{rs}$.
2. Symmetry*:* $\pi_{sr} = \pi_{rs}$, *for all* $r, s \in \{1, \dots, g\}$.
3. Maximum proximity*:* $\pi_{rs} = \delta_1 \Leftrightarrow r = s$, *for all* $r, s \in \{1, \dots, g\}$.
4. Monotonicity*:* $\pi_{rs} \succ \pi_{rt}$ *and* $\pi_{st} \succ \pi_{rt}$, *for all* $r, s, t \in \{1, \dots, g\}$ *such that* $r < s < t$.

**Definition 2 ([2]).** *An OPM* $\pi : \mathscr{L} \times \mathscr{L} \longrightarrow \Delta$ *is* totally uniform *if* $\pi_{r\,(r+t)} = \pi_{s\,(s+t)}$ *for all* $r, s, t \in \{1, \dots, g-1\}$ *such that* $r + t \leq g$ *and* $s + t \leq g$.

We now introduce the notion of scoring function on an OQS. These functions assign numerical scores to the linguistic terms of OQSs satisfying two basic conditions.

**Definition 3.** *Given an OQS* $\mathscr{L} = \{l_1, \dots, l_g\}$, *a* scoring function on $\mathscr{L}$ *is a function* $S : \mathscr{L} \longrightarrow \mathbb{R}$ *satisfying the following conditions for all* $r, s \in \{1, \dots, g\}$*:*

1. $S(l_r) < S(l_s) \Leftrightarrow r < s$.
2. *If* $\pi$ *is the totally uniform OPM on* $\mathscr{L}$, *then there exists* $d > 0$ *such that* $S(l_r) = S(l_1) + (r-1) \cdot d$.

In the following proposition we introduce four scoring functions that take into account the perceptions about the ordinal proximities between the linguistic terms of the OQSs. $S_b$ is based on the comparison between each linguistic term and the **b**est linguistic term, $l_g$. $S_w$ is based on the comparison between each linguistic term and the **w**orst linguistic term, $l_1$. $S_{bw}$ is the average of the two previous scoring functions. $S_a$ is based on the comparison between each linguistic term and **a**ll linguistic terms.

**Proposition 1.** *Let $\mathscr{L} = \{l_1, \dots, l_g\}$ be an OQS equipped with an OPM $\pi : \mathscr{L} \times \mathscr{L} \longrightarrow \Delta$ and $\rho(\delta_k) = k$.*

*The functions $S_b$, $S_w$, $S_{bw}$, $S_a : \mathscr{L} \longrightarrow \mathbb{R}$ defined as*

$$S_b(l_r) = h - \rho(\pi_{rg}) \qquad S_w(l_r) = \rho(\pi_{r1}) - 1$$
$$S_{bw}(l_r) = \frac{S_b(l_r) + S_w(l_r)}{2} = \frac{h + \rho(\pi_{r1}) - \rho(\pi_{rg}) - 1}{2}$$
$$S_a(l_r) = \frac{(g+2)\cdot(g-1)}{2} + \sum_{s<r} \rho(\pi_{sr}) - \sum_{s>r} \rho(\pi_{rs})$$

*are scoring functions on $\mathscr{L}$.*

Note that $S_b(l_1) = S_w(l_1) = S_{bw}(l_1) = 0$, $S_b(l_g) = S_w(l_g) = S_{bw}(l_g) = h-1$ and, consequently, $S_b(l_r), S_w(l_r), S_{bw}(l_r) \in [0, h-1]$ for every $r \in \{1, \dots, g\}$. The four scoring functions can be easily normalized in the unit interval.

**Acknowledgements**
The financial support of the Spanish *Ministerio de Ciencia e Innovación* (project PID2021-122506NB-I00) is acknowledged.

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# Missing Data completing employing a Hybrid Fuzzy-Rough Approach in a Posture Detection System

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A new hybrid system (FRSystem) [1] based on fuzzy and rough sets [2] has been developed, as an improvement of camera-based systems [3]. Due to the fact that FRSystem is dedicated to elderly monitoring, it is desirable to extend it by the possibility to mine incomplete data. Therefore, a new method, based on the usage of a new K measure of knowledge approach and a new method for computing maximal consistent blocks from incomplete data, was proposed. A methodology was evaluated in the following stages. The knowledge measure-only approach was based on searching for the most similar objects among the same decision class, as the missing value object, with possibly the highest knowledge measure value. Whereas in the second approach, we include the idea of computing maximal consistent blocks. Thus, the blocks with the highest probabilistic approximation are selected, from among whom the objects with the biggest Knowledge measure are chosen.

The proposed hybrid approach was tested in a posture detection system, on data with 5%, 25%, and 50% of missing values. The best results for the interval-valued fuzzy model with K measure were obtained using the geometric and arithmetic means as aggregations and for the rough-fuzzy model using the maximum as aggregation in the inference process. The usage of the knowledge measure minimized the uncertainty due to imprecision and incompleteness of data.

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# Nonparametric tests for fuzzy data

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Most popular statistical tests are constructed with fairly specific assumptions concerning the underlying population distribution, such as normal, exponential, etc. If we also assume that we are just interested in the values of parameters of these distributions, or when examining differences between populations we can relate them only to some of their characteristics, like means or variances, it is often possible to derive optimal tests in the Neyman-Pearson framework. However, these tests can be sensitive to violations of the assumptions of the model, and in reality, these assumptions are often not met. Meanwhile, any conclusions drawn from such tests are as valid as the assumptions made. So if we have doubts about whether the assumptions hold or if there is not enough information to assess their validity, nonparametric statistical methods should be used.

This type of situation occurs in particular when we are faced with imprecise or vague observations, so often in real-life data, especially human ratings based on opinions or associated with perceptions. Random fuzzy numbers (also known as fuzzy random variables) constitute a useful model which allows grasping both randomness, associated with the data generation mechanism, and fuzziness, connected with data imprecision. However, the derivation of nonparametric tests for fuzzy data is by no means straightforward. Moreover, it cannot be done by directly generalizing nonparametric tests for real-valued data to fuzzy data. Indeed, many effective nonparametric tests are based on ranks that are easily determined and which give the distribution-free character to the statistical procedure. Meanwhile, we cannot rank directly observations in fuzzy samples since their realizations are not linearly ordered. For the same reason, we cannot use signs or series, which are techniques that make a nonparametric toolbox. But, even worse, these are not all the inconveniences associated with fuzzy data analysis. Difficulties in subtraction and division of fuzzy numbers, no tractable models for the distribution of fuzzy random variables, and no Central Limit Theorems that can be applied directly in calculations with random fuzzy numbers - all these disadvantages impede a simple and natural generalization of statistical tools applied for reasoning with crisp data, in particular, in hypotheses testing. Of course, the need to overcome difficulties releases creative potential. In the case of tests for fuzzy data, the distance-based approach combined with the bootstrap [1] or permutation tests [2–6].

In this contribution, we consider two ideas: the distance-based approach combined with permutation tests and the use of the credibility index [7] measuring the extent of the degree of the dominance relationship for each pair of random variables. Both approaches lead to some effective statistical nonparametric procedures that admit fuzzy data.

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# Generalized integrals applied to scientometric indices

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As it is well-known, see [5], the two prominent fuzzy integrals of Choquet and Sugeno are closely related to scientometry. Indeed, the Choquet integral of scientometric record (arranged in a nonincreasing way) with respect to the counting measure gives the number of all author's citations. On the other hand, the Sugeno integral of scientometric record with respect to the counting measure gives the $h$-index of the author. Thus, both integrals provide complementary summaries (although there are taken with respect to the same function and the same measure). Several extensions are provided in [2].

Recently, authors in [1] developed a concept of conditional aggregation operators. We consider a nonempty set $X$ and a $\sigma$-algebra $\Sigma$ of subsets of $X$. Hereafter, $\Sigma^0 = \Sigma \setminus \{\emptyset\}$. By $\mathbf{F}$ we denote the set of all $\Sigma$-measurable nonnegative bounded functions on $X$.

**Definition 1.** Let $E \in \Sigma^0$. A map $\mathsf{A}(\cdot|E)\colon \mathbf{F} \to [0,\infty]$ is said to be a *conditional aggregation operator w.r.t. $E$* (*CAO*, for short) if it satisfies the following conditions:

($C1$) $\mathsf{A}(f|E) \le \mathsf{A}(g|E)$ for any $f, g \in \mathbf{F}$ such that $f(x) \le g(x)$ for all $x \in E$;
($C2$) $\mathsf{A}(\mathbb{1}_{E^c}|E) = 0$, where $E^c = X \setminus E$.

The concept of the generalized survival function [1] and the generalized level measure [4] are important for definition of the generalized integrals (of Choquet- as well as Sugeno-type). We just recall the *generalized level measure* of $f \in \mathbf{F}$ w.r.t. a family $\mathscr{A} = \{\mathsf{A}(\cdot|E)\colon E \in \Sigma\}$ of CAOs with $\mathsf{A}(\cdot|\emptyset) = \infty$ and a monotone measure $\mu : \Sigma \to [0,\infty]$ being defined by

$$\mu_{\mathscr{A}}(f,t) = \sup\left\{\mu(E)\colon \mathsf{A}(f|E) \ge t,\, E \in \Sigma^0\right\}, \quad t \in [0,\infty).$$

Setting $\mathscr{A}^{\inf} = \{\mathsf{A}^{\inf}(\cdot|E)\colon E \in \Sigma\}$ one obtains the level measure

$$\mu_{\mathscr{A}^{\inf}}(f,a) = \mu(\{f \ge a\}), \quad a \in [0,\infty).$$

In the contribution we restrict our attention to the Choquet and Sugeno-like operators [3] of the form

$$\mathbf{Ch}_{\mathscr{A}}(f,\mu) = \int_0^\infty \mu_{\mathscr{A}}(f,t)\,\mathrm{d}t,$$
$$\mathbf{Su}_{\mathscr{A}}(f,\mu) = \sup_{t\in[0,\infty)} t \wedge \mu_{\mathscr{A}}(f,t),$$

being the generalizations of the standard Choquet and Sugeno integrals (for $\mathscr{A} = \mathscr{A}^{\mathrm{inf}}$). We provide several properties of these operators w.r.t. the properties of $\mathscr{A}$. Applying the results for the scientific records, we discuss possibilities of eliminating weaknesses of the $h$-index and related scientometric indices.

**Acknowledgements**
The work was supported by the Slovak Research and Development Agency under the contract No. APVV-21-0468, the grant VEGA 1/0657/22, and the internal scientific grant vvgs-pf-2022-2143.

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# Pseudo-uninorms with Archimedean underlying functions

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A concept of uninorms, proposed by Yager and Rybalov in 1996 [1], unifies the theory of t-norms and t-conorms in a common ground. Since then, the uninorms were studied thoroughly both in theoretical study and terms of application in fuzzy theory, decision making, neural networks etc. However, in some cases, it is useful to drop the axiom of commutativity of uninorms (see for instance [2, 3]). This leads us to the definition of a pseudo-uninorm. Similarly as in the case of the uninorms, each pseudo-uninorm is isomorphic to a t-psuedo-norm on the square $[0, e]^2$ and to a t-pseudo-conorm on $[e, 1]^2$, which are jointly called the underlying functions. In this conference contribution we would like to focus on the cases, when both underlying functions are continuous and Archimedean and thanks to the result of Hölder [4] also commutative and thus can be regarded as a t-norm and a t-conorm. Finally our aim is to provide a total characterization of these pseudo-uninorms. In this study we continue in works of Fodor et al. [5, 6], where a characterization of uninorms with continuous Archimedean underlying functions was shown.

**Acknowledgements**
The work was supported from grants VEGA 1/0036/23, VEGA 2/0142/20, APVV-18-0052 and APVV-20-0069.

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# A generalization of Bi-polar OWA operators to the continuous case

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Mesiar et al. [4, 5] introduced bi-polar OWA operators (BIOWA) as a natural extension of OWA operators which were introduced by Yager [7]. Bi-polarity enables us to use both, positive and negative evaluations in a decision-making process. However, the BIOWA, as introduced in [4, 5], are discrete with a finite domain.

As pointed out by Narukawa et al. [6], when the domain of aggregation is uncountable, a generalization is necessary. In Jin et al. [3], a generalization of OWA operators to an uncountable domain was proposed. Based on the ideas in [3], we will propose BIOWA operators for an uncountable domain. We will show how the BIOWA on an uncountable domain (continuous BIOWA) are connected to the bi-polar Choquet integral [2] and bi-polar capacities [1]. Another result concerns the constraints that are necessary to adopt when constructing the continuous BIOWA.

**Acknowledgements**

The work was supported from the VEGA grant agency, grant No. 2/0142/20 and by the Science and Technology Assistance Agency under contract No. APVV-18-0052.

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# The non-linear impact of the scaling factor $\alpha$ on the outcomes of Semi-Supervised Fuzzy C-Means

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Semi-Supervised Fuzzy C-Means model (SSFCM) originally proposed in [3] is a popular fuzzy clustering algorithm that has been studied and extended by many researchers (see e.g. [1]). It adapts a classical, unsupervised Fuzzy C-Means to handle partial supervision in a form of categorical labels assigned to a part of all available observations. The partial supervision mechanism is implemented by establishing an arbitrary one-to-one mapping between clusters and classes.

The extent to which the supervised observations influence the outcomes of the model is controlled with a single hyperparameter $\alpha > 0$ called a scaling factor. The objective function formalizing the task of SSFCM is composed of two components: an unsupervised one, and the supervised one. The scaling factor weighs the supervised component.

[2] analyze descriptions of $\alpha$ proposed in the literature and conclude that none of them considers the scaling factor in the context of the main outcome of SSFCM, which is the estimated memberships matrix (containing degrees of membership of all observations to all clusters). The descriptions interpret $\alpha$ only in the context of the objective function, which leads to an implicit conclusion that the impact of $\alpha$ on the outcomes of the modeling is directly proportional. [2] argue that such a conclusion is incorrect and propose a novel explanation of the scaling factor that exhibits a non-linear impact of $\alpha$ on the estimated membership degrees. In this work, we extend [2] and present the derivation of the novel explanation with reference to the optimization algorithm. We show the consequences of modifying the functional form of the non-linear impact of the scaling factor on the outcomes of SSFCM presented in this explanation. The main findings are illustrated with simulations and experiments for real-life data.

**Acknowledgements**

Kamil Kmita and Katarzyna Kaczmarek-Majer received funding from Small Grants Scheme (NOR/SGS/BIPOLAR/0239/2020-00) within the research project: “Bipolar disorder prediction with sensor-based semi-supervised Learning (BIPOLAR)”.

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# Some approaches for divergence measures between fuzzy sets

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A comparison between two fuzzy sets can be done by different comparison measures, by considering different points of view. In some cases these are measures of the equality degree (e.g. similarity measures) and in other cases the difference degree (e.g. dissimilarity measures, divergence measures and distance measures) [1]. Among these measures, divergences appear as a good alternative in some cases [4].

Let $(X, T, S)$ be a triple, where $X$ is the universe and $T$ and $S$ are any t-norm and t-conorm, respectively. A map $D : \mathscr{F}(X) \times \mathscr{F}(X) \to \mathbb{R}$ is a divergence measure with respect to $(X, T, S)$ iff for all $A, B \in \mathscr{F}(X)$, $D$ satisfies the following conditions:

(1) $D(A, A) = 0$;
(2) $D(A, B) = D(B, A)$;
(3) $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$, for all $C \in \mathscr{F}(X)$, where the union and intersection are defined by means of $S$ and $T$, respectively [2].

Therefore we introduced a more general concept for locality. Let $(X, T, S)$ be a triple with $X$ a finite universe and $T$ and $S$ any t-norm and t-conorm, respectively. Divergence measure $D$ associated to $(X, T, S)$ is local if and only if $D(A, B) = \sum_{x \in X} h_x(A(x), B(x))$, where $\{h_x\}_{x \in X}$ is a family of maps from $[0, 1] \times [0, 1]$ into $\mathbb{R}$ such that, for any $x \in X$ and $a, b, c \in [0, 1]$, there is:

(i) $h_x(a, a) = 0$, for all $a \in [0, 1]$,
(ii) $h_x(a, b) = h_x(b, a)$, for all $a, b \in [0, 1]$,
(iii) $h_x(a, b) \geq \max(h_x(S(a, c), S(b, c)), h_x(T(a, c), T(b, c)))$ for all $a, b, c \in [0, 1]$.

This concept has been generalized in three directions [3]:

(1) We have considered the triple $(X, T, S)$ with a triangular norm $T$ and its dual t-conorm $S$ as integral parts of the underlying universe $X$, that represent the method for constructing intersections and unions, or on the other hand, the logic for the particular model.
(2) We have considered different maps $h_x$ for the points (or ranges of points) in $X$ we can stress the specific aspects of a particular model and thus obtain more accurate information when specifying the measure of difference. It is one of the advantages of the proposed attitude.

(3) We have introduced and justified a new class of divergence measures between two fuzzy subsets, the $S$-local divergences, which is constructed by using a triangular conorm $S$ instead of the sum, i.e. a map $D$ is an $S$-local divergence measure if and only if $D(A, B) = S_{x \in X} h_x(A(x), B(x))$. The family of local divergences is generalized to the family of $S$-local divergences based on arbitrary triangular conorm $S$. They are obtained by combining the divergence at particular points of the universe with "distances" among them using a triangular conorm.

This generalization allows us to manage a larger class of divergence measures and use them to define new classes of fuzziness measures. New examples of divergence measures can be obtained when applying aggregation function on them. Some additional conditions need to be satisfied to preserve properties for divergence measures will be presented.

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# Building Trust in Machine Learning: The Role of Explainable AI

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Recently, machine learning helps decision-making in many aspects of our lives. We train models to predict the future in banking, telecommunications, insurance, industry, and many other fields. We use many types of models for this, from simple tree model-type structures to complex ones such as random forests or neural networks. Using complex models increases their predictive power, but unfortunately, we usually do not know how they are calculated. Reducing the interpretability of models can lead to a lack of understanding of the results obtained, which can negatively affect decisions. Can we trust these predictions?

The approach to predictive modeling has changed in recent years. Much more attention is now paid to explanations that show what influences the model's decision. This is enforced by the EU's General Data Protection Regulation (GDPR) [2]. Explainable artificial intelligence (XAI) is one of the latest approaches that aim to provide results that can be understood by a human expert. It attempts to explain "black–box" models for data scientists, i.e., neural networks, random forest, or xgboost.

We want to explain the predictions of the machine learning model. To do this we need explanatory methods, i.e., algorithms that generate explanations [1]. An explanation is a way of linking the model's predictions to values that describe an entity in a human-understandable way. Explanations can be divided into global and local explanations. Global explanations are those that explain which features are important, how important they are, and how they interact with each other. Local explanations, on the other hand, are those that show the change in decisions or contributions of variables for a single observation.

One of the most popular algorithms to explain machine learning is the LIME algorithm introduced by Ribeiro [5]. This technique explains the prediction of any classifier by fitting a weighted linear model on the observations similar to the observation of interest. Another popular algorithm is Shapey values [4]. This local technique is based on the coalitional game theory method; variables are treated as players, which can be in different coalitions. The contribution of a variable is an average over its all coalitions.

An important element of explainable artificial intelligence is domain knowledge. Fuzzy logic has been successfully used to incorporate expert knowledge into decision–making processes. This technique can be useful in making artificial intelligence more comprehensible, as it can help combine natural language rules with numerical decisions. Incorporating fuzzy logic techniques may increase the transparency and interpretability of machine learning models, making them more explainable [3]. Furthermore, it could help build confidence in machine learning and ensure that models are making the right decisions.

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# Consistency of linguistic summaries

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Motivated by [2] we discuss some aspects of the interpretability of fuzzy linguistic summaries. They are natural language sentences that describe and recap the set of numerical data that cannot be easily grasped. Interpretability can be investigated on two level approaches: the sentence level and the summary level. Here, we focus on the second one and on the notion of consistency. A summary can be called consistent if the properties of non-contradiction and double negation are satisfied. Both of them are considered for the protoform "$Q$ $B$y's are $A$", where $Q$ is a linguistic quantifier, $B$ a qualifier, and $A$ a summarizer [1, 3].
Non-contradiction property is fulfilled when two sentences with contradictory terms have complementary truth values, while the double negation property refers to the case when two contraries applied to the protoform do not change its truth values. Note that the double negation $D$ of a protoform $P$ is defined as follows:

$$D(P) = \neg QB \text{ y's are } \neg A.$$

To have a consistent summary, $\neg A$ (a negation of $A$) has to satisfy

$$\neg A(x) = 1 - A(x), \quad x \in [0, 1],$$

and the quantifier $\neg Q$ must meet

$$1 - \neg Q(x) = \neg Q(1 - x), \quad x \in [0, 1]. \tag{1}$$

In this contribution, we investigate possible fuzzy negations satisfying (1) and discuss another negation operators $\neg$ of the summarizer than a classical negation.

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# Exploring the Diversity of Honeycomb Polygonal Chains Triplets

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The study of polygonal chains on a honeycomb grid has recently gained significant interest due to their potential applications in various scientific fields. In particular, the problem of comparing and aggregating such structures has been a topic of great interest.

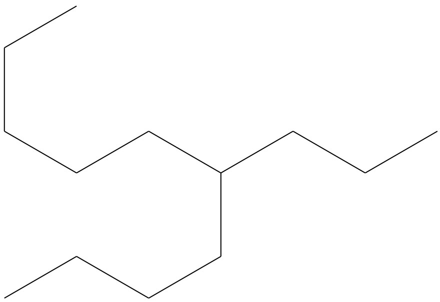

**Fig. 1.** Example of a honeycomb polygonal chains triplet structure

In this work, we propose a method for comparing and aggregating honeycomb polygonal chains triplets consisting of three non-intersecting chains on a honeycomb grid (cf. structure shown in Figure 1). We introduce a novel metric to quantify the similarity and dissimilarity between the triplets based on the numerical representation of such structures. Furthermore, we consider the issue of the invariance of the structure with respect to its rotations.

We will consider the following metric for honeycomb polygonal chains.

$$\mathrm{d}(B, B') = \sum_{k=0}^{n} (n - k + 1) \cdot |B(k) - B'(k)|. \tag{1}$$

In the equation (1), we assume that $B$ and $B'$ have the same length. We will assume that the shorter one has zeros added at the end for binary sequences with different lengths. Then, we will use such a distance function to calculate dissimilarity for honeycomb polygonal chains triplets comparing corresponding polygonal chains with respect to rotations.

We also present a novel method for aggregating multiple triplets into a single structure, which preserves the essential features of the original triplets. The aggregation function is obtained by applying the above metrics in constructing centroids and medoids.

We conclude by discussing the potential applications of our methods in various fields and suggest directions for future research.

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# Position of “A few” and “Several” in Graded Peterson’s Hexagon of Opposition

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We will focus on the construction of the graded Peterson’s hexagon of opposition in fuzzy natural logic with new forms of intermediate quantifiers “A few” and “Several”. The main goal will be to mathematically formulate a new definition of the contradictory property. Next, we will discuss the properties that are satisfied for the new forms of quantifiers.

This contribution will focus on the construction of the graded Aristotle’s and Peterson’s hexagons of opposition in fuzzy natural logic as an extension of graded Aristotle’s square of opposition. Aristotle’s square of opposition ([7]) was studied in many publications. Note that Aristotle’s square of opposition is fulfilled with presupposition only. The reader can find further details about the necessity of presupposition in [2]. The reader can find further details about the necessity of presupposition in [2]. In [3, 6], the authors draw a crucial distinction between the “classical” Aristotelian square of opposition and the “modern” duality one based on the concepts of inner and outer negation. Another extension using intermediate quantifiers, which in terms of meaning are just among the classic quantifiers, was developed first by Thompson by adding the inter-mediate quantifiers “Almost all” and “Many” (see [5]). The final form was later proposed by Peterson (see [4]), who introduced and philosophically explained the position of the basic five intermediate quantifiers (“All”, “Almost all”, “Most”, “Many” and “Some”) in the square. Graded version of both squares was syntactically and also semantically analyzed in [2].

Béziau in [1] suggested extending the square of opposition into a hexagon. This technically means adding two new formulas **U** and **Y** that are defined as the disjunction of the two top corners of the square and conjunction of the two bottom corners:

$$\mathrm{U} = \mathbf{AE}: \text{ All or No } B \text{ are } A \tag{1}$$

$$\mathbf{Y} = \mathrm{IO}: \text{ Some but Not All } B \text{ are } A. \tag{2}$$

Another motivation of this research is the introduction of new forms of fuzzy intermediate quantifiers based on the graded Peterson’s square of opposition and to construct graded Peterson’s hexagon of opposition. The new construction leads us to the idea of using natural language to describe new properties.

*A few people are vaccinated against COVID-19 and a few people are not vaccinated against COVID-19*

**Acknowledgements**
OP PIK CZ.01.1.02/0.0/0.0/17147/0020575 AI-Met4Laser: Consortium for industrial research and development of new applications of laser technologies using artificial intelligence methods (10/2020 -6/2023), of the Ministry of Industry and Trade of the Czech Republic.

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# Relational modifiers for L-valued mappings

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In a process of decision making or at a product evaluation we can observe statements that are not comparable, i.e. statements on a product design versus those on its functionality, statements on a location of a building versus those on its energy demands, etc. Such statements can lead us to use a bounded lattice $L$ as a model for these kinds of statements.

If in the previous consideration we assume a finite set of evaluators, we obtain a mapping from a finite (perhaps possible to generalize to a discrete) domain $X$ to the lattice. Moreover, we assume that there is a total order in this domain. Hence we obtain a lattice-valued fuzzy set $A : X \to L$. This fuzzy set corresponds to a collection of single independent statements, one by each evaluator. However, there may exist some mutual relations among the evaluators, reflecting the measure of mutual influence. For this we use a binary fuzzy relation $R$ on $X$.

In a situation, when the evaluators after their initial statements are allowed to observe evaluations of the others and subsequently modify their own evaluation, we obtain a modifier, i.e a mapping $M : L^X \to L^X$. To reflect the relations among evaluators we study these modifiers in the following form: For a lattice-valued fuzzy set $A$, the relation $R$ on $X$ and $\alpha \in [0, 1]$ the modified lattice-valued fuzzy set is

$$^{\mathscr{A}}M_R^{\alpha}(A) = \mathscr{A}\{A(y) \in L; R(x, y) \geq \alpha\},$$

where $\mathscr{A}$ is an aggregation function on $L$.

We study properties of these modifiers depending on $\mathscr{A}$ and $R$, namely preserving monotony. This extends the attitude used in [1] and [2], where only comparable statements are assumed.

**Acknowledgements**
The work was supported by Grant No.1/0150/21 provided by Slovak grant agency VEGA (the second author).

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# Classifiers Aggregation method based on the Distributivity Law (CADL)

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As a result of the huge increase in computer processing power and the ability to store and process large amounts of data, computers have become an integral part of our world. We are witnessing a real explosion of databases in terms of their number and volume.
The simplicity of construction and acceptable cost have led to data collection systems being used in almost all areas of life. One of the main purposes of data collection is to discover the dependencies hidden within it. Modern methods of data mining are provided by modern multivariate statistics, where classification methods are of particular practical importance. Ensemble (hybrid) methods or multiple classifiers are known as learning algorithms that train a set of classifiers and combine them to achieve the best prediction accuracy [1], [2], [3]. The most fundamental concepts of ensemble methods consist of two main stages which are the production of multiple base classifier models and their combination via aggregation. Aggregation functions proved to be an effective tool in many application areas [4]. Basically, they refer to the calculations performed on a dataset to get a single number that accurately represents the underlying data.
There are also many approaches to using domain knowledge and improving the quality of data mining models (see e.g. [5]). Thus, the use of aggregation functions can be treated as a way to use domain knowledge to improve the quality of classifiers.
Classification and detection of various types of attacks in a computer network is currently an important field of scientific research. Any attempt to corrupt the confidentiality, integrity, and accessibility of information is called a network attack. Today, there are many different attacks on information systems. Intrusion detection systems have been developed to prevent these attacks. Intrusion detection systems are designed to take precautions against the risk of attack on the network, they monitor all network traffic and identify suspicious situations in incoming and outgoing connections.

This study, being a continuation of the research with a novel hybrid approach (see [6], [7]) aims to apply and evaluate the usefulness of an ensemble classifier to detect network intrusion threats on comprehensive datasets. The approach uses a distributivity law that appropriately aggregates the underlying classifiers (acronym: CADL method). Considerations include the results of experiments conducted on the UNSW-NB 15 dataset, a collection of network packets exchanged between hosts.

A five-fold cross-validation technique using the Scikit Learn tool was used to evaluate the performance of the classifiers on the dataset. We present the results of comparing our hybrid algorithm resulting from aggregation (triangular norms and means) with the distributivity equation of selected classification algorithms (Multilayer Perceptron Network, k Nearest Neighbours, and Naive Bayes) with themselves on the raw data.

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# Inflation expectation analysis with evolving membership functions

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Inflation expectations are a key variable for Central Banks that have adopted inflation targeting. At the same time, it is an unobservable economic variable, estimated mainly based on linguistic surveys among consumers. Therefore the use of fuzzy analysis, in this case, is particularly justified but still rarely found in the literature. In this specific application, a time-dependent procedure for constructing membership functions is crucial and we use the fuzzy linguistic summarization with evolving membership functions [1]. After all, the perception of what level of inflation is high and what level is low changes over time due to the economic conditions surrounding us at any given time.

In this study, we apply the fuzzy linguistic summarization to analyze inflation expectations. One of the main advantages of fuzzy linguistic summaries is their human consistency. At the same time, one of the main challenges when generating fuzzy linguistic descriptions using type-1 fuzzy sets is the proper definition of membership functions that describe the linguistic terms related to the considered attributes ([2], [3]). To reflect the changing nature of economic variables, the definitions of linguistic terms are updated over time using statistical modeling and results of the Business and Consumers Surveys. The study covers European inflation-targeting countries and their ability to influence inflation expectations through the outcome of the messages of the monetary policy councils' meetings. An attempt was made to describe the relationship between CB messages tone (sentiment) and customer inflation expectation in the dynamic economic situation and inflation perception. We compare the static and dynamic approaches to calculate the membership values that describe the linguistic terms related to inflation expectations with and without including survey data. An additional variable considered is the transparency of the CBs under consideration. Using linguistic summaries makes it possible to supplement traditional statistical analysis with interpretable results, an essential part of the transparency of central banks performing inflation expectations analyses.

**Acknowledgements**

Aleksandra Rutkowska is supported by the National Science Centre, Poland grant No.2020/37/B/HS4/02611

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# Regression models for optimizing the aluminum extrusion process under isothermal conditions

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The motivation for the paper topic was the need to optimize an industrial aluminum hot extrusion process due to obtaining isothermal conditions of the extruded profile which directly affects the high quality of the product. For this purpose, the use of the Finite Element Method (FEM) model was developed, and then the regression model was based on its results (minimum temperature variations at the exit of the die). Numerical FEM simulations were carried out in accordance with a specially designed experimental plan based on the Greco-Latin square where the reference industrial case was selected, the research area was defined and three independent variables with their variability levels were selected. The temperature gradient obtained at the exit of the profile from the die was selected as the process isothermal criterion, assuming the smallest temperature gradient was the most favorable. Numerical simulations were carried out using specialized QForm software that can perform calculations of material flow as thermally and mechanically coupled using CFD (Computer Fluid Dynamic) which is a requirement in constitutive modeling of hot extrusion processes. The FEM simulation results based on ingot back temperature, ingot front temperature, and extrusion speed, all with three levels of variability, were used to develop the regression model. The temperature of the extruded profile at the exit of the tool was investigated and its gradient (differences between the minimum and maximum value) was presumed as an output (dependent) variable. The obtained results of numerical experiments were initially analyzed in terms of their significance for the tested output variable showing the statistical significance of all three variables. Then, the obtained results were used to check several parametric regression models (both with or without interactions), and their results were confronted with industrial data for the tested process. This comparison showed that the variables that ensure optimal process isotherm conditions went beyond the manufacturer's practice from which the industrial data was derived.

Keeping in mind that industrial data abounds with multiple types of uncertainty, it seems reasonable to turn from parametric to non-parametric regression [1, 2]. Therefore, we will consider models based on kernel methods. We will also take into account the latest developments related to the fuzzy transform, which turns out to be very effective in a variety of data analysis tasks (see, e.g., [3–6]).

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# Generalized mixture $\mathscr{D}$-mean

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The so-called moderate deviation function $\mathscr{D} : \mathbf{I}^2 \to \bar{\mathbf{R}}$, where $\mathbf{I}$ is a closed interval $\mathbf{I} = [a, b] \subset \bar{\mathbf{R}}$, $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$, can be defined as a function which fulfills following properties [1, 3, 4, 6]:

(i) for every $x \in \mathbf{I}$, $\mathscr{D}(x, \cdot) : \mathbf{I} \to \bar{\mathbf{R}}$ is increasing (not necessarily strictly);

(ii) for every $y \in \mathbf{I}$, $\mathscr{D}(\cdot, y) : \mathbf{I} \to \bar{\mathbf{R}}$ is decreasing (not necessarily strictly);

(iii) $\mathscr{D}(x, y) = 0$ if and only if $x = y$, $x \in \mathbf{I}$, $y \in \mathbf{I}$.

The function $\mathscr{G}^{\mathscr{D}} : \mathbf{I}^n \times \mathbf{I} \to \bar{\mathbf{R}}$ given by $\mathscr{G}^{\mathscr{D}}(\mathbf{x}, y) = \sum\limits_{i=1}^{n} \mathscr{D}(x_i, y)$ is called a global moderate deviation function and the mapping given by $U^{\mathscr{D}} : \mathbf{I}^n \to \mathbf{I}$,

$$U^{\mathscr{D}}(\mathbf{x}) = \frac{1}{2}\left(\sup\left\{y \in \mathbf{I} \,\middle|\, \sum_{i=1}^{n} \mathscr{D}(x_i, y) < 0\right\} + \inf\left\{y \in \mathbf{I} \,\middle|\, \sum_{i=1}^{n} \mathscr{D}(x_i, y) > 0\right\}\right)$$

is called a $\mathscr{D}$-mean, with standard conventions $\sup\{y \in [a, b] | y \in \emptyset\} = a$ and $\inf\{y \in [a, b] | y \in \emptyset\} = b$, and it is an idempotent symmetric aggregation function [6].

The contribution introduces the extension of the theory of the construction of aggregation functions using the so-called generalized mixture global moderate deviation function $Gg\mathscr{G}^{\mathscr{D}} : \mathbf{I}^n \times \mathbf{I} \to \bar{\mathbf{R}}$, which is defined for any $\mathbf{x} \in \mathbf{I}^n$ and $y \in \mathbf{I}$ by

$$Gg\mathscr{G}^{\mathscr{D}}(\mathbf{x}, y) = \sum_{i=1}^{n} g_i(x_i) \cdot \mathscr{D}_i(x_i, y),$$

where $g_i : \mathbf{I} \to ]0, \infty[$ are continuous weighting functions associated with the input values $x_i$, and $\mathscr{D}_i$ are moderate deviation functions; $i = 1, 2, \ldots, n$. The vector $\mathbf{g} = (g_1, \ldots, g_n)$ is called a weighting vector.

The mapping $GgU^{\mathscr{D}} : \mathbf{I}^n \to \mathbf{I}$ given by

$$GgU^{\mathscr{D}}(\mathbf{x}) = \frac{1}{2}\left(\sup\left\{y \in \mathbf{I} \Big| \sum_{i=1}^{n} g_i(x_i) \cdot \mathscr{D}_i(x_i, y) < 0\right\} + \inf\left\{y \in \mathbf{I} \Big| \sum_{i=1}^{n} g_i(x_i) \cdot \mathscr{D}_i(x_i, y) > 0\right\}\right)$$

is called a generalized mixture $\mathscr{D}$-mean, with standard conventions $\sup\{y \in [a,b] | y \in \emptyset\} = a$ and $\inf\{y \in [a,b] | y \in \emptyset\} = b$.

However, its monotonicity can be violated, which means that in general it cannot be considered as an aggregation function [2, 5].

Our ambition is to present sufficient conditions to maintain the monotonicity of the generalized mixture $\mathscr{D}$-mean.

**Acknowledgements**

The work has been supported by the Slovak Scientific Grant Agency VEGA no. 1/0150/21.

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# The learning method of regression using interval-valued spline functions

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Logistic regression is used in a variety of fields, including machine learning, most fields of medicine, and the social sciences. Logistic regression can be used to predict the risk of developing a given disease (e.g. diabetes, ischemic heart disease), based on the observed characteristics of the patient (age, sex, body mass index, results of various blood tests, etc.), see for example [1].

The basic idea behind logistic regression is to use a linear combination of explanatory variables and a set of regression coefficients that are model-specific but the same for all trials. Linear predictor function *f(i)* for a specific data point $i$ is written as:

$$f(i) = \beta_0 + \beta_1 x_{1,i} + \cdots + \beta_m x_{m,i},$$

where $\beta_0, \ldots, \beta_m$ are regression coefficients indicating the relative influence of a particular explanatory variable on the result. The basic learning algorithm consisted in updating the values of the weighting factors scalar betas, depending on the error between the predicted value and the actual value decision for a given record.

The proposal for a new approach concerns the encoding method and the update mechanism weighting factors that take the form of a series of splines, and disjoint values coded by compartments. By default, after the training stage, the weights for individual beta coefficients receive a value real.

In the new approach, one weighting factor can have multiple scalar values that are strictly assigned input values contained in a given range. An example representation of a certain beta value is proposed in Fig. 1. A simplified diagram of

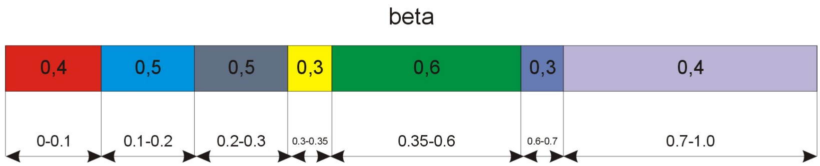


**Fig. 1.** Weighting factor with multiple scalar values

the learning mechanism, using a spline of certain functions with an assigned value, indexed by interval matching, is shown in Fig. 2.

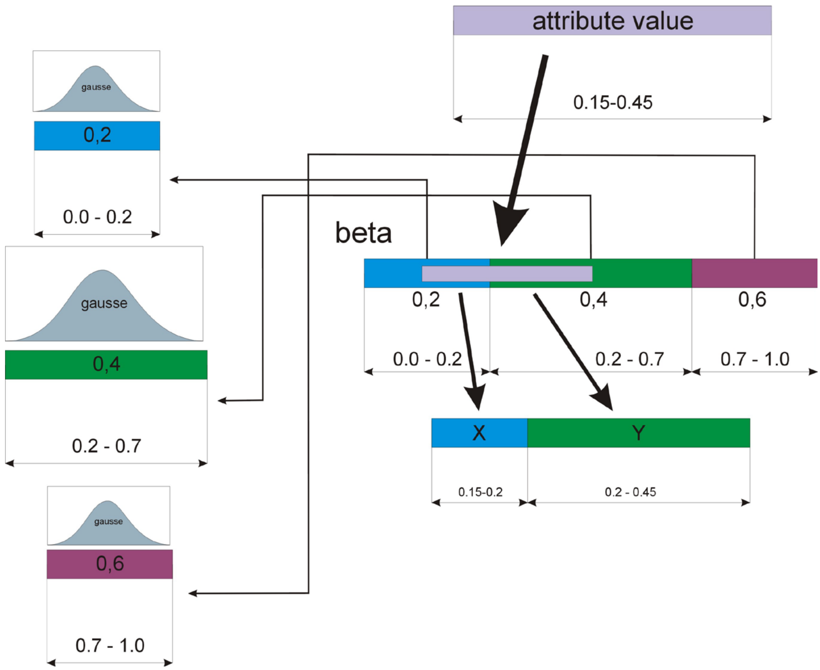


**Fig. 2.** A new mechanism for selecting and updating weighting factors in the learning process

We also present an algorithm for determining the weighted beta coefficients for excitation from a given input/attribute, and also proposals for weight aggregation methods in a new form, because in many practical issues, it is a necessary process, e.g. in federated learning (e.g. [2]).

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# Fuzzy Rule comparison system and its application to select the best method for the Generalized Modus Ponens

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Zadeh introduced in 1965 the generalized modus ponens [1], which has been widely used to create inference systems which deal with imprecise information. This systems use if-then rules, which have fuzzy sets as antecedents and consequents. To resolve the GMP Zadeh also proposed a mechanism called the compositional rule of inference [2]. More methods have been developed ([3][4][5][6]) with the same purpose. With this variety of methods, and the parameters required for each one of them, several choices have to be made to select them for any given application ([7][8]).

The main goal of this contribution is to establish a rule comparison measure for fuzzy rules composed of fuzzy sets as antecedents and fuzzy sets as consequents. This measure employs different indices between fuzzy sets (for each pair of antecedents and consequents) and then via aggregation functions a comparison value for the rules is obtained.

The secondary goal is to present a decision making system for the GMP. This system will use the defined comparison measure to select the best method and its corresponding parameters to resolve the GMP.

**Acknowledgements**
The work has was supported by research project PID2019-108392GB-I00 (AEI/10.13039 /501100011033) of the Agencia Estatal de Investigación and by the projects CNPq (301618/2019-4,305805/2021-5) and FAPERGS (19/2551-0001660-3). Asier Urio-Larrea has a predoctoral grant from Santander-UPNA 2021/22.

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# Choquet integral as a Substitute for Multiplication in Vision Transformers

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The Choquet integral is a common aggregation function for applications that have interaction among input data. This lead to diverse applications such as decision-making [3], brain-computer interfaces [7], and classification [5]. In addition to those applications, a handful of generalizations emerged which in multiple cases have improved previous results that used the Choquet integral. Some examples of those generalizations are the d-Choquet integral [1], the $CF_1F_2$-integral [4], and the Vector Choquet integral (VCI) [2].

On the other hand, the Choquet integral can also be applied to algorithms, like Transformers [6]. Currently, this type of algorithm have a one single aggregation function to the self-attention mechanism, which uses the product sum as aggregation function.

The objective of the present study is to enhance the performance of Transformers and analyze the results, by integrating Choquet's integral as an aggregation function within the self-attention mechanism of the algorithm.

By doing so, it is hypothesized that the resulting model will exhibit improved efficiency and effectiveness.

**Acknowledgements**
This work was supported by Navarra de Servicios y Tecnología (NASERTIC), Agencia Estatal de Investigación (Government of Spain) under the project PID2019-108392GB-I00 (AEI/10.13039/501100011033), FAPERGS/Brazil (Proc. 19/2551-0001660-3), and CNPq/Brazil (301618 /2019-4, 305805/2021-5).

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# Fuzzy Multi-Criteria Decision Analysis: Myth or Reality?

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Fuzzy Multicriteria Decision Analysis (FMCDA) is one of the key directions within fuzzy decision analysis and Fuzzy Set theory. More than two dozen fuzzy extensions of ordinary MCDA methods and their variations based on different types of FS have been developed and used within various applications [1, 2]. At the same time, the development of FMCDA has the place [mainly] in breadth. However, an in-depth analysis of the properties of FMCDA from theoretical and methodological points of view is actually missing. The title of this abstract refers to both the fundamental of FMCDA and the practice of its application. Consider just a few examples.

1. Within MCDA, there is the so-called Basic Axiom (BA): if alternative $A$ is Pareto superior to alternative $B$, then within the MCDA method $M$, $B$ cannot exceed (have a higher rank) $A$. However, this axiom is violated in FMCDA [3, 4].
2. In most FMCDA publications, when assessing functions of Fuzzy Numbers (FNs), the dependence of the quantities used is not taken into account, which leads to a significant overestimation of the output FNs and affects decision-making [3, 4, 6, 7].
3. Despite there are several dozen methods for ranking of FNs, there is a problem with the use of fuzzy ranking methods in FMCDA. The most popular ranking method in FMCDA is Centroid Index, CI (or Center of Gravity), although its advantage over others is more than doubtful [3, 6, 7]. In [6], some novel fuzzy ranking methods have been suggested, which take into account the dependence of ranked FNs, however, their use within FMCDA requires further exploration.
4. The most popular (or the only) approach to assessing functions of FNs within FMCDA is the approximate one based on the use and propagating Triangular (TrFNs) and Trapezoidal FNs (TpFNs) through all computations. However, there can be significant distinctions in output results on ranking/sorting alternatives if (for the comparison) the standard fuzzy arithmetic or proper determining functions of FNs is implemented [7].
5. Within FMCDA practice, the so-called *presumption of model adequacy* [7] is implemented: I have developed a model and I propose its application. At the same time, the problems mentioned above (including those in para 1, 2, and 4) are not investigated.
6. One of the well-known concepts (or maybe, the key one) of fuzzy decision-making is as follows: decision-making in the fuzzy environment should be inherently fuzzy. In [5] an approach to implementation of such a conception has been suggested

(Fuzzy Multicriteria Acceptability Analysis), However, it needs further exploration.
7. To date, thousands of articles have been published within FMCDA based on the novel FS [1, 2]. As far as I know, there is no discussion of the key problems, mentioned above (including para 1 and 4).

All the problems mentioned in paras 1-7 are of great theoretical and methodological significance and require further research, including both shortcomings of the existing methodology, and clarification of the inevitable problems of decision-making under conditions of uncertainty/fuzziness.

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